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Link polynomials related to the new braid group representations

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Abstract. By introducing a modified diagonal matrix **h**, the properties of Markov moves are examined for the non-standard representations of the braid group associated with the fundamental representations of A_n, B_n, C_n and D_n. It is shown that link polynomials can be constructed from those braid group representations. The skein relations of link polynomials are obtained explicitly.

1. Introduction

Recently there has been a great deal of interest in the study of braid group representations and their related topological invariants (link polynomials) [1-8] due to the fact that they are tightly concerned with two-dimensional statistical mechanics, quantum integrable systems and conformal field theory [9-14]. It is known that link polynomials can be derived from certain braid group representations via a Markov trace [2, 15]. A sequence of new braid group representations called non-standard representations of braid groups are obtained [16] under the constraints of weight conservation conditions [15]. However, whether polynomials can be derived from them was not clear then. In a previous letter [17] we constructed the polynomial for a simple sort of non-standard representation of braid group associated with fundamental representation of A_n.

In this paper, introducing a modified (non-positive definite) diagonal matrix **h**, we show that link polynomials can be constructed from all the known non-standard representations of braid group associated with fundamental representation of A_n, B_n, C_n and D_n. In the next section we derive the constraints by Markov move I with the modified matrix **h**, and discuss the cases of A_n, B_n, C_n and D_n concretely. In section 3 we give the main results of non-standard representations of braid group in some more convenient notations. In section 4 we examine Markov moves I and II respectively for the non-standard representations of braid group, and derive the link polynomials from them explicitly. In section 5 we give some remarks and discussions.

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2. The constraints by Markov move I with modified h

Link polynomials are functionals of topological equivalent classes of links. They can be constructed from certain braid group representations since any link can be regarded as a closed braid and vice versa. The closed braids from equivalent braids or inequivalent ones, but mutually transformed by Markov moves, give the same link. Thus for a given non-trivial representation of braid group, whether a polynomial can be constructed from it depends on whether a Markov trace can be defined properly.

In order to obtain link polynomials from the non-standard representations of braid group, we introduce a modified diagonal matrix **h**

$$h=(h^a_b) \qquad h^a_b := \delta'_a q^{4\Lambda_a\rho+\Delta_a}\delta^a_b \tag{1}$$

where $\delta'_a=\pm 1$, $\Delta_a=4\Lambda_a\varepsilon$, δ^a_b is the Kronecker delta, ρ is half the sum of all positive roots of a Lie algebra and Λ_a ($a\in I\subset\mathbb{Z}$) are weight vectors of an irreducible representation of the Lie algebra.

For the modified **h**, the property of Markov move I (see [2] and [15]) $\mathrm{tr}(\mathbf{H}A_1A_2)=\mathrm{tr}(\mathbf{H}A_2A_1)$ requires that

$$(\delta'_a\delta'_b q^{4(\Lambda_a+\Lambda_b)(\rho+\varepsilon)}-\delta'_c\delta'_d q^{4(\Lambda_c+\Lambda_d)(\rho+\varepsilon)})S^{ab}_{cd}=0 \tag{2}$$

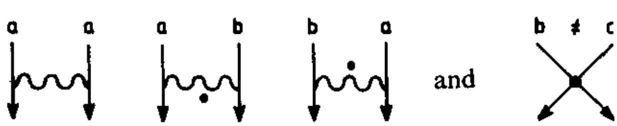
where S^{ab}_{cd} are the elements of the **S**-matrix of braid group representation [2, 15]. This leads to $S^{ab}_{cd}=0$ unless

$$\Lambda_a+\Lambda_b=\Lambda_c+\Lambda_d \tag{3}$$

and

$$\delta'_a\delta'_b=\delta'_c\delta'_d. \tag{4}$$

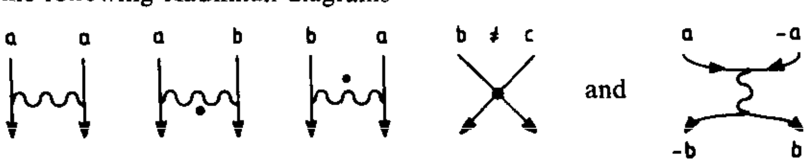
In the case of fundamental representation of A_n, the non-vanishing contributions to the **S**-matrix determined by weight conservation condition (3) [15] are the following Kauffman [3] diagrams



(5)

where the set of labels is $I=\{n, n-2, \ldots, -n+2, -n\}$. Obviously, (4) holds identically for the non-vanishing elements of the **S**-matrix.

In the cases of fundamental representations of B_n, C_n and D_n, the non-vanishing contributions to the **S**-matrix are determined without much difficulty [15]. They are the following Kauffman diagrams



(6)

where the sets of labels are $I=\{2n, 2n-2, \ldots, -2n+2, -2n\}$ for B_n, $I=\{2n-1, 2n-3, \ldots, 1, -1, \ldots, -2n+3, -2n+1\}$ for C_n and D_n. Equation (4) holds for the first four diagrams of (6). For the last diagram of (6), equation (4) becomes

$$\delta'_a\delta'_{-a}=\delta'_{-b}\delta'_b. \tag{7}$$

Because (7) should be satisfied for any a and b in the set of labels, it requires that

$$\delta'_a = \delta'_{-a} \qquad \forall a \in l \tag{8a}$$

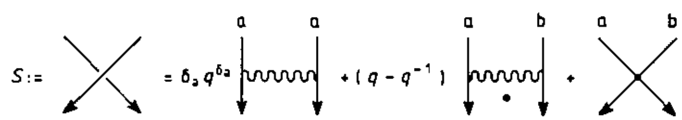
or

$$\delta'_a = -\delta'_{-a} \qquad \forall a \in l. \tag{8b}$$

3. Non-standard representations of a braid group (main results)

Starting from the structure of braid group representation which is determined by the weight conservation condition [15], one can obtain the braid group representation explicitly by solving the spectral parameter-independent Yang–Baxter equation (YBE) directly. If all coefficients of non-vanishing Kauffman diagrams are not assumed to be independent of labels (certainly the transposition symmetry is still adopted due to the prime star invariance of polynomials [15]), a sequence of new solutions of parameter-independent YBE will be obtained. This gives the so-called non-standard representations of braid group [16]. Using the notation $\delta_a = \pm 1$, we can write the results of [16] as follows.

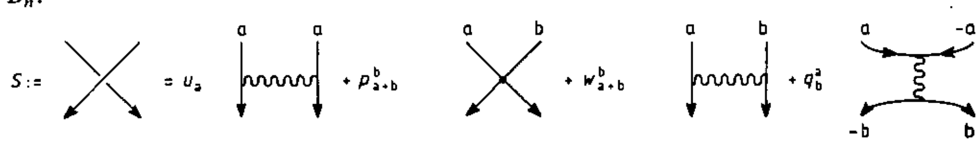
A_n:



(9)

where δ_a takes +1 or −1 arbitrarily, different arrangement of the two values in $\{\delta_a\}$ corresponds to different representation of braid group. Evidently the standard case is that every δ_a equals 1 instead of −1 or just vice versa.

B_n:



(10)

where

$$u_a = \delta_a q^{\delta_a - \delta_{a,0}} \qquad \delta_{-a} = \delta_a \qquad \delta_0 = 1$$

$$p^b_{a+b} = 1 \qquad w^b_{a+b} = q - q^{-1} \qquad \text{for } a + b \neq 0$$

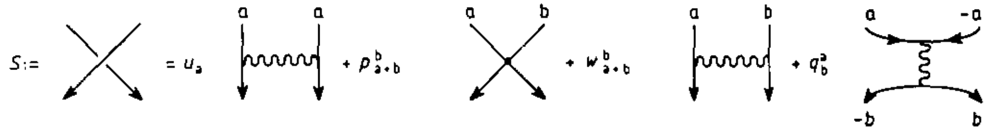
$$p^b_0 = (u_b)^{-1} \qquad w^b_0 = (q - q^{-1})\left(1 - u_a \prod_{c=-b}^{b} u_c^{-1}\right) \tag{11}$$

$$q^a_b = (-1)^{(a+b)/2+1}(q - q^{-1})u_a^{1/2}u_b^{1/2} \prod_{c=a}^{b} u_c^{-1} \qquad (a < b)$$

$$q^a_b = 0 \qquad (a > b).$$

If all $\delta_a = 1$, it gives the standard representation.

C_n and D_n:



(12)

where

$$u_a=\delta_a q^{\delta_a} \qquad \delta_{-a}=\delta_a$$

$$p^b_{a+b}=1 \qquad w^b_{a+b}=q-q^{-1} \qquad \text{for } a+b\neq 0$$

$$p^b_0=u_b^{-1}$$

$$w^b_0=\begin{cases}(q-q^{-1})\left(1+u_b u_1^{-1}\prod\limits_{c=-b}^{b} u_c^{-1}\right) & \text{for } C_n\\ (q-q^{-1})\left(1-u_b u_1\prod\limits_{c=-b}^{b} u_c^{-1}\right) & \text{for } D_n\end{cases} \tag{13}$$

$$q^a_b=\begin{cases}-\dfrac{|ab|}{ab}u_1^{(|ab|/ab-1)/2}(q-q^{-1})u_a^{1/2}u_b^{1/2}\prod\limits_{c=a}^{b}u_c^{-1} & \text{for } C_n\\ -u_1^{(1-|ab|/ab)/2}(q-q^{-1})u_a^{1/2}u_b^{1/2}\prod\limits_{c=a}^{b}u_c^{-1} & \text{for } D_n\end{cases} \qquad (a<b)$$

$$q^a_b=0 \qquad (a>b).$$

When $\delta_a=1$ for any $a\in l$, they become standard cases.

4. Construction of link polynomials

All of the non-standard representations of braid group were found under the weight conservation condition. Thus, once (4) is satisfied by non-vanishing elements of the **S**-matrix, Markov move I is guaranteed. Moreover δ'_a and Δ_a in (1) should be determined by the property of Markov move II. Now we examine the properties of Markov moves for the non-standard representations of braid group.

4.1. Examination of Markov moves

Comparing the representations of braid group given in section 3 with (8), one can easily find that if

$$\delta'_a=\delta_a \tag{14}$$

equation (4) is satisfied whatever the cases of B_n, C_n or D_n. Therefore the property of Markov move I is satisfied for the cases of A_n, B_n, C_n and D_n when δ'_a in (1) is just the notation δ_a in the non-standard representations of braid group.

It is not very difficult to prove that if

$$\Delta_{a-2}-2=\Delta_a-(\delta_{a-2}+\delta_a) \qquad a\in l \tag{15}$$

for A_n and B_n, and

$$\begin{aligned} &\Delta_{a-2}-2=\Delta_a-(\delta_{a-2}+\delta_a) \qquad (a\neq 1)\\ &\Delta_{-1}-4=\Delta_1-4\delta_1 \qquad \text{for } C_n \\ &\Delta_{-1}=\Delta_1 \qquad \text{for } D_n \end{aligned} \tag{16}$$

for C_n and D_n, the property of Markov move II [2] will be satisfied, i.e.

$$\begin{aligned} &\sum_b S_{ab}^{ab} h_b^b=\tau \qquad \text{independent of } a \\ &\sum_b (S^{-1})_{ab}^{ab} h_b^b=\bar{\tau} \qquad \text{independent of } a. \end{aligned} \tag{17}$$

It is easy to calculate that $\tau=q^{n+\Delta_n+\delta_n}$, $q^{2n-1+\Delta_{2n}+\delta_{2n}}$, $q^{2n+\Delta_{2n-1}+\delta_{2n-1}}$, and $q^{2n-2+\Delta_{2n-1}+\delta_{2n-1}}$; $\bar{\tau}=q^{-n+\Delta_{-n}-\delta_{-n}}$, $q^{-2n+1+\Delta_{-2n}-\delta_{-2n}}$, $q^{-2n+\Delta_{-2n+1}-\delta_{-2n+1}}$ and $q^{-2n+2+\Delta_{-2n+1}-\delta_{-2n+1}}$, for A_n, B_n, C_n and D_n respectively.

4.2. *Link polynomials*

Once the Markov trace is defined concretely, the link polynomial can be calculated explicitly. The formula is the same as that in [2, 15]

$$P(A)=(\tau\bar{\tau})^{-(m-1)/2}\left(\frac{\bar{\tau}}{\tau}\right)^{e(A)/2}\phi(A) \qquad A\in\mathscr{B}_m \tag{18}$$

where

$$\phi(A)=\mathrm{tr}(Hg(A)) \tag{19}$$

$$H=\prod^m\otimes h \tag{20}$$

but $\mathbf{h}$ given by (1) is different.

The eigenvalues of the $\mathbf{S}$-matrix (9) is found to be q and $-q^{-1}$. So the reduction relation of braid group representation $g_i=I^{(1)}\otimes\ldots\otimes I^{(i-1)}\otimes S\otimes I^{(i+2)}\otimes\ldots\otimes I^{(m)}$ is

$$(g_i-q)(g_i+q^{-1})=0. \tag{21}$$

After solving the recursive relations (15) of Δ_a for A_n, we obtain from (18) and (21) the following skein relation for polynomials of the A_n case:

$$q^{\mu}P_+-(q-q^{-1})P_0-q^{-\mu}P_-=0 \tag{22}$$

where we have adopted a notation

$$\mu:=\sum_{b\in I}\delta_b. \tag{23}$$

It is found that the $\mathbf{S}$-matrices of cases B_n, C_n and D_n have three distinct eigenvalues: $\lambda_1=q$, $\lambda_2=-q^{-1}$ for B_n, C_n and D_n but $\lambda_3=q^{-\mu+1}$ for B_n, $\lambda_3=-\delta_1 q^{-\mu-\delta_1}$ for C_n and $\lambda_3=\delta_1 q^{-\mu+\delta_1}$ for D_n. Then the reduction relations of braid group representations of those cases can be written down. After solving the recursive relations of Δ_a (equations (15) and (16)) for those cases, we obtain the following cubic skein relations in an analogous way:

$$B_n:\quad q^{2(\mu-1)}P_{+2}-(q^{\mu}-q^{\mu-2}+1)P_{+1}-(q^{-\mu}-q^{-\mu+2}+1)P_0+q^{-2(\mu-1)}P_{-1}=0 \tag{24}$$

$$C_n: \quad q^{2(\mu+\delta_1)}P_{+2}-(q^{\mu+\delta_1+1}-q^{\mu+\delta_1-1}-\delta_1)P_{+1} \\ +\delta_1(q^{-\mu-\delta_1-1}-q^{-\mu-\delta_1+1}-\delta_1)P_0-\delta_1 q^{-2(\mu+\delta_1)}P_{-1}=0 \tag{25}$$

$$D_n: \quad q^{2(\mu-\delta_1)}P_{+2}-(q^{\mu-\delta_1+1}-q^{\mu-\delta_1-1}+\delta_1)P_{+1} \\ -\delta_1(q^{-\mu+\delta_1-1}-q^{-\mu+\delta_1+1}+\delta_1)P_0+\delta_1 q^{-2(\mu-\delta_1)}P_{-1}=0 \tag{26}$$

where the notation of (23) has been used.

5. Remarks and discussion

In the above we have shown that link polynomials can be defined from the so-called non-standard representations of braid group. The key step is introducing an appropriate diagonal matrix **h** so that the Markov trace can be defined. Actually the Markov trace defined by standard trace of matrix with a non-positive definite diagonal matrix can be considered as that defined by a supertrace with a positive definite diagonal matrix, i.e.

$$\Phi(A)=\mathrm{str}(Hg(A)) \qquad A\in\mathscr{B}_m \tag{27}$$

where $\mathrm{str}(M)=\mathrm{tr}(\mathscr{H}M)$, $\mathscr{H}=\Pi^m\otimes\eta$, $\eta^a_b=\delta_a\delta^a_b$ while $H=\Pi^m\otimes h$, $h^a_b=q^{4\Lambda_a(\rho+\varepsilon)}$. As we showed in [14], ε is the sum of some roots.

One may notice that (23) means

$$\mu=\mathrm{tr}\,\eta. \tag{28}$$

For the standard case, η is a unit matrix and then μ is the dimension of the matrix. The skein relations (22) and (24)-(26) depend on the integer μ, so each skein relation for link polynomials corresponds to one standard representation and a series of non-standard representations having the same μ and δ_1 (the latter only for the cases of C_n and D_n). The skein relation (22) of the A_n case is equivalent to that constructed from the vertex models associated with $\mathrm{gl}(m|n)$ [18].

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